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Journal of Zankoy Sulaimani
Part-A- (Pure and Applied Sciences)

Generalized Semistrongly Convex Fuzzy Sets

Pishtiwan O. Sabir

Department of Mathematics, Faculty of Science and Education Sciences, University of Sulaimani, Sulaimani, Kurdistan-Region, Iraq
E-mail: pishtiwan.sabir@gmail.com, pishtiwan.sabir@univsul.edu.iq

Article info

Original:
25January 2016
Revised:
21March 2016
Accepted:
21 April 2016
Published online:
20 September 2016

Key Words:
Fuzzy sets Fuzzy convex sets,Semistrongly convex fuzzy sets

Abstract

The intention of this work is to study various aspects on the concepts of generalized convex fuzzy sets, generalized strongly convex fuzzy sets, and generalized semistrongly convex fuzzy sets. Precisely has investigated the properties and relations among them.

Introduction, Definitions and Notations

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [1] in 1965 and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets [2, 3-4, 5]. Zadeh's work continued research and scholars extensively studied fuzzy convexity such as [6-9, 10, 11-13, 14-16].

Let A be a nonempty subset of the universal set X. Then its grade of membership, μ_A, maps from X into $\{0,1\}$. The concept of fuzzy sets is a generalization of the ordinary sets, maps X into the closed interval $[0,1]$defined by $\tilde{A}=\{(x,\mu_{\tilde{A}}(x)) : x\in X\}$. Frequently, we will write $\mu(x,\tilde{A})$ instead of $\mu_{\tilde{A}}(x)$ and will denoted the fuzzy power set on $X$by $\mathcal{F}(X)$.

Let$\tilde{A},\tilde{B}\in\mathcal{F}(X)$. Then $\tilde{A}$ is a superset of $\tilde{B}$ is defined by $\mu(x,\tilde{A})\geq\mu(x,\tilde{B})$, for every $x\in X$. A fuzzy set $\tilde{A}$ on $\mathcal{R}$ is convex if and only if $\mu(y,\tilde{A})\geq\mu(y_1,\tilde{A})\wedge\mu(y_2,\tilde{A})$, where $y=\beta y_1+(1-\beta)y_2,\quad y_1,y_2\in\mathcal{R}$ and $\beta\in[0,1]$.

Let$\tilde{A}_i\in\mathcal{F}(X)$, $i\in I$ (I is a nonempty index set). Then the standard fuzzy intersection of $\tilde{A}_i$, $\cap_i\tilde{A}_i$, is defined by $inf_x\,\mu(x,\tilde{A}_i)=\wedge_x\,\mu_{\tilde{A}_i}(x)$; the standard fuzzy union of $\tilde{A}_i$, $\cup_i\tilde{A}_i$, is defined by $sup_x\,\mu(x,\tilde{A}_i)=\vee_x\,\mu_{\tilde{A}_i}(x)$; and the complement of $\tilde{A}_i$, $\neg\tilde{A}_i$, is defined by $\mu(x,\neg\tilde{A}_i)=1-\mu(x,\tilde{A}_i)$, for all $x\in X$. The union of a fuzzy set $\tilde{A}_i$ and $\neg\tilde{A}_i$ should not necessarily give the whole X. Also, the intersection between $\tilde{A}_i$ and its complement $\neg\tilde{A}_i$ is not necessarily give the empty set.

One of the basic notions of fuzzy subsets is the Zadeh's extension principle. This extension first implied in [1] in an elementary presentation and was finally in [17] and [18] are presented. This principle provides a method for extending crisp mathematical notions to fuzzy quantities as the arguments of the function. Let $g{:}\,A_1\times A_2\times\ldots\times A_n\to B$ given by $y=g(a_1,a_2,\ldots,a_n)$ and $\tilde{A}_i\in\mathcal{F}(X_i)$for $i=1,2,\ldots n$. Here the set $\tilde{C}=g(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n)$ is defined by

$$\tilde{C} = \left\{\left(y, \bigvee\left(\mu(a_1, \tilde{A}_1)\wedge\mu(a_2, \tilde{A}_2)\wedge \dots \wedge\mu(a_n, \tilde{A}_n)\right)\right) : y = g(a_1, a_2, \dots, a_n) \text{and } (a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \dots \times A_n\right\}$$

Let $\tilde{A}', \tilde{A}'' \in \mathcal{F}(X)$. If we denote the extended addition and multiplication by $\tilde{+}$and $\tilde{\cdot}$, respectively, then by the Zadeh's principle, one obtains

$$\mu(x, \tilde{A}'\tilde{+}\tilde{A}'') = \bigvee_{x_1, x_2 \mid x = x_1 + x_2} \left(\mu(x_1, \tilde{A}')\wedge\mu(x_2, \tilde{A}'')\right)$$

and

$$\mu(x, x_2 \tilde{\cdot} \tilde{A}'') = \mu(xx_2^{-1}, \tilde{A}''), x_2 > 0.$$

As a generalization of Zadeh's fuzzy set, the concept of interval valued fuzzy set was presented by Gorzalczany [19] and introduced for the first time by Turksen [20]. Let $\mu(\tilde{A}, x)$ and $\mu(x, \tilde{A})$ denotes for lower fuzzy set and upper fuzzy set about $\mu^{\tilde{A}}$, respectively, then the mapping $\mu^{\tilde{A}}_{(x)}: A \to [\mu(\tilde{A}, x), \mu(x, \tilde{A})]$ is called an interval valued fuzzy set on A and frequently we shall call generalized fuzzy sets. All generalized fuzzy sets on the universal set X are denoted by $GFS(X)$.

Let $\mu^{\tilde{A}} \in GFS(\mathcal{R})$. Then $\mu^{\tilde{A}}$ is called generalized convex fuzzy set [21], if for any $x_1, x_2 \in \mathcal{R}$and $\beta \in [0,1] \coloneqq I^{1]}_{[0}$, we have $\mu^{\tilde{A}}_{(\beta x_1 + (1-\beta)x_2)} \geq \mu^{\tilde{A}}_{(x_1)} \wedge \mu^{\tilde{A}}_{(x_2)}$. Equivalently, $\mu^{\tilde{A}}$ is a generalized convex fuzzy set if and only if its lower fuzzy set and upper fuzzy set about $\mu^{\tilde{A}}$ are convex fuzzy set.

Let $\mu^{\tilde{A}} \in GFS(X)$and $\beta_1, \beta_2 \in I^{1]}_{[0}$. We define ${}^{+}\mu^{\tilde{A}}_{[\beta_1, \beta_2]}$ and ${}^{-}\mu^{\tilde{A}}_{(\beta_1, \beta_2)}$, respectively, as the crisp set of all elements of the universal set X that belongs to lower fuzzy set about $\mu^{\tilde{A}}$ at least to the degree β_1 and upper fuzzy set about $\mu^{\tilde{A}}$ at least to the degree β_2; the ordinary set that contains all elements of the universal set whose membership grades in the given lower set are greater than but do not include the specified value of β_1 and membership grades in the given upper set are greater than but do not include the specified value of β_2.

Main Results

This section, gives results concerning generalized fuzzy sets, fuzzy convex sets, and related topics.

Definition 1: Let $\mu^{\tilde{\mathcal{F}}} \in GFS(\mathcal{R})$. Then $\mu^{\tilde{\mathcal{F}}}$ is called a generalized strongly convex fuzzy set if

$$\mu^{\tilde{\mathcal{F}}}_{(\mu\lambda + (1-\mu)\gamma)} > [\mu(\tilde{\mathcal{F}}, \lambda)\wedge\mu(\tilde{\mathcal{F}}, \gamma), \mu(\lambda, \tilde{\mathcal{F}})\wedge\mu(\gamma, \tilde{\mathcal{F}})]$$

for every $(\lambda \neq \gamma)$ $\lambda, \gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$ and $\mu \in I^{1)}_{(0}$.

Definition 2: Let $\mu^{\tilde{\mathcal{F}}} \in GFS(\mathcal{R})$. Then $\mu^{\tilde{\mathcal{F}}}$ is called a generalized semistrongly convex fuzzy set if

$$\mu^{\tilde{\mathcal{F}}}_{(\mu\lambda + (1-\mu)\gamma)} > [\mu(\tilde{\mathcal{F}}, \lambda)\wedge\mu(\tilde{\mathcal{F}}, \gamma), \mu(\lambda, \tilde{\mathcal{F}})\wedge\mu(\gamma, \tilde{\mathcal{F}})]$$

for every $\lambda, \gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$, $\mu(\tilde{\mathcal{F}}, \lambda) \neq \mu(\tilde{\mathcal{F}}, \gamma)$, $\mu(\lambda, \tilde{\mathcal{F}}) \neq \mu(\gamma, \tilde{\mathcal{F}})$ and $\mu \in I^{1)}_{(0}$.

Theorem 3: Let $\mu^{\tilde{\mathcal{F}}}, \mu^{\tilde{G}} \in GFS(\mathcal{R})$. If $\mu^{\tilde{\mathcal{F}}}$ and $\mu^{\tilde{G}}$ are both generalized strongly convex fuzzy set (resp. generalized convex fuzzy set), then standard fuzzy intersection of $\mu^{\tilde{\mathcal{F}}}$ and $\mu^{\tilde{G}}$ is generalized strongly convex fuzzy set (resp. generalized convex fuzzy set).

Proof: We will prove only for the part that $\mu^{\tilde{\mathcal{F}}}$ and $\mu^{\tilde{G}}$ are both generalized strongly convex fuzzy set. Assume there exists $(\lambda \neq \gamma)$ $\lambda, \gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}\cap\tilde{G}}_{[0]}$and $\mu \in I^{1)}_{(0}$. By hypothesis, we have

$$\mu^{\tilde{\mathcal{F}}\cap\tilde{G}}_{(\mu\lambda + (1-\mu)\gamma)} = [\mu(\tilde{\mathcal{F}} \cap \tilde{G}, \mu\lambda + (1-\mu)\gamma), \mu(\mu\lambda + (1-\mu)\gamma, \tilde{\mathcal{F}} \cap \tilde{G})]$$

this implies that

$$\mu^{\tilde{\mathcal{F}}\cap\tilde{G}}_{(\mu\lambda + (1-\mu)\gamma)} > [\mu(\tilde{\mathcal{F}}, \mu\lambda + (1-\mu)\gamma)\wedge\mu(\tilde{G}, \mu\lambda + (1-\mu)\gamma), \mu(\mu\lambda + (1-\mu)\gamma, \tilde{\mathcal{F}})\wedge\mu(\mu\lambda + (1-\mu)\gamma, \tilde{G})]$$

from strongly convex definition we get

$$\mu^{\tilde{\mathcal{F}}\cap\tilde{G}}_{(\mu\lambda+(1-\mu)\gamma)} > \left[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma)\wedge\mu(\tilde{G},\lambda)\wedge\mu(\tilde{G},\gamma), \mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{F})\wedge\mu(\lambda,\tilde{G})\wedge\mu(\gamma,\tilde{G})\right]$$

Finally, the result follows from the equality

$$\begin{aligned}&\left[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{G},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma)\wedge\mu(\tilde{G},\gamma), \mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\lambda,\tilde{G})\wedge\mu(\gamma,\tilde{F})\wedge\mu(\gamma,\tilde{G})\right]\\ &= \left[\mu(\tilde{\mathcal{F}}\cap\tilde{G},\lambda)\wedge\mu(\tilde{\mathcal{F}}\cap\tilde{G},\gamma), \mu(\lambda,\tilde{\mathcal{F}}\cap\tilde{G})\wedge\mu(\gamma,\tilde{\mathcal{F}}\cap\tilde{G})\right]\end{aligned}$$

Remark 4: Let $\mu^{\tilde{\mathcal{F}}_i} \in GFS(\mathcal{R})$be any family of generalized strongly convex fuzzy set (resp. generalized convex fuzzy set) for all $i \in I$, where I is a nonempty index set. Then standard fuzzy intersection, $\mu^{\cap_{i\in I}\tilde{\mathcal{F}}_i}$, is a generalized strongly convex fuzzy set (resp. generalized convex fuzzy set) on $\mathcal{R}$.

Remark 5: Let $\mu^{\tilde{\mathcal{F}}}, \mu^{\tilde{G}} \in GFS(\mathcal{R})$. If $\mu^{\tilde{\mathcal{F}}}$ and $\mu^{\tilde{G}}$ are generalized semistrongly convex fuzzy set, then standard fuzzy intersection of $\mu^{\tilde{\mathcal{F}}}$ and $\mu^{\tilde{G}}$ is not necessarily a generalized semistrongly convex fuzzy set. Furthermore, the standard fuzzy intersection on infinitely generalized semistrongly convex fuzzy set is not necessarily a generalized semistrongly convex fuzzy set. The following example covers the remark;

$$\mu(\tilde{\mathcal{F}},\xi) = \begin{cases}0.3, & \text{when } \xi = 0\\ 0.5, & \text{when } \xi = 1\\ 1.0, & \text{when } \xi \neq 0 \text{ and } \xi \neq 1\end{cases}$$

$$\mu(\xi,\tilde{\mathcal{F}}) = \begin{cases}0.1, & \text{when } \xi = -1\\ 0.6, & \text{when } \xi = 1\\ 1.0, & \text{when } \xi \neq -1 \text{ and } \xi \neq 1\end{cases}$$

$$\mu(\tilde{G},\xi) = \begin{cases}0.3, & \text{when } \xi = 1\\ 0.5, & \text{when } \xi = 2\\ 1.0, & \text{when } \xi \neq 1 \text{ and } \xi \neq 2\end{cases}$$

$$\mu(\xi,\tilde{G}) = \begin{cases}0.1, & \text{when } \xi = -2\\ 0.6, & \text{when } \xi = 2\\ 1.0, & \text{when } \xi \neq 2 \text{ and } \xi \neq -2\end{cases}$$

It is easy to verify that for $\lambda = 0$, $\gamma = 2$ and $\mu = 0.5$, we have

$$\mu^{\tilde{\mathcal{F}}\cap\tilde{G}}_{(\mu\lambda+(1-\mu)\gamma)} = [0.3, 0.6] = \mu^{\tilde{\mathcal{F}}\cap\tilde{G}}_{(\lambda)} \wedge \mu^{\tilde{\mathcal{F}}\cap\tilde{G}}_{(\gamma)}$$

Theorem 6: Let $\mu^{\tilde{\mathcal{F}}} \in GFS(\mathcal{R})$ be a generalized convex fuzzy set. If there exists $\pi \in I^{1)}_{(0}$ for every $(\lambda \neq \gamma)$ $\lambda, \gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$ implies that

$$\mu^{\tilde{\mathcal{F}}}_{(\pi\lambda+(1-\pi)\gamma)} > \left[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma), \mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})\right],$$

then$\mu^{\tilde{\mathcal{F}}}$ is generalized strongly convex fuzzy set on $\mathcal{R}$.

Proof: By way of contradiction, assume that there exists $(\lambda \neq \gamma)$ $\lambda, \gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$, $\alpha \in I^{1)}_{(0}$ for which

$$\mu^{\tilde{\mathcal{F}}}_{(\alpha\lambda+(1-\alpha)\gamma)} \leq \left[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma), \mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})\right]$$

On the other hand we have,

$$\mu^{\tilde{\mathcal{F}}}_{(\alpha\lambda+(1-\alpha)\gamma)} \geq \mu^{\tilde{\mathcal{F}}}_{(\lambda)} \wedge \mu^{\tilde{\mathcal{F}}}_{(\gamma)}$$

So that,

$$\mu^{\tilde{\mathcal{F}}}_{(\alpha\lambda+(1-\alpha)\gamma)} = \left[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma), \mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})\right]$$

Let $\nu \in I^{1)}_{(0}$ such that $\alpha = \pi\nu + (1-\pi)\nu$and $\lambda = \nu\lambda + (1-\nu)\gamma = \gamma$. Since $\mu^{\tilde{\mathcal{F}}} \in GFS(\mathcal{R})$ is a generalized convex fuzzy set, we can get

$$\begin{aligned}\mu(\tilde{\mathcal{F}},\lambda) &= \mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma) \geq \mu(\tilde{\mathcal{F}},\lambda) \wedge \mu(\tilde{\mathcal{F}},\gamma) = \mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma) \wedge \mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma)\\ &= \mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma)\end{aligned}$$

and

$$\begin{aligned}\mu(\gamma,\tilde{\mathcal{F}}) &= \mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}}) \geq \mu(\lambda,\tilde{\mathcal{F}}) \wedge \mu(\gamma,\tilde{\mathcal{F}}) = \mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}}) \wedge \mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}})\\ &= \mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}})\end{aligned}$$

and

$$\mu(\tilde{\mathcal{F}},\gamma)=\mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma)\geq\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma)=\mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma)\wedge\mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma)$$
$$=\mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma)$$

and

$$\mu(\lambda,\tilde{\mathcal{F}})=\mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}})\geq\mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})=\mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}})\wedge\mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}})$$
$$=\mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}})$$

By a simple calculationon the equations of$\alpha,\lambda,$ and γ defined above, we obtain$\mu(\tilde{\mathcal{F}},\alpha\lambda+(1-\alpha)\gamma)=\mu(\tilde{\mathcal{F}},\pi\lambda+(1-\pi)\gamma)$and $\mu(\alpha\lambda+(1-\alpha)\gamma,\tilde{\mathcal{F}})=\mu(\pi\lambda+(1-\pi)\gamma,\tilde{\mathcal{F}})$. So, the inequality of the theoremgives that

$$\begin{aligned}\mu^{\tilde{\mathcal{F}}}_{(\alpha\lambda+(1-\alpha)\gamma)}=\mu^{\tilde{\mathcal{F}}}_{(\pi\lambda+(1-\pi)\gamma)}\\ &>[\mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma)\wedge\mu(\tilde{\mathcal{F}},\nu\lambda+(1-\nu)\gamma),\mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}})\\ &\wedge\mu(\nu\lambda+(1-\nu)\gamma,\tilde{\mathcal{F}})]\\ &\geq\left[\left(\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma)\right)\wedge\left(\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma)\right),\left(\mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})\right)\\ &\wedge\left(\mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})\right)\right]\end{aligned}$$

Hence,

$$\mu^{\tilde{\mathcal{F}}}_{(\alpha\lambda+(1-\alpha)\gamma)}>[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma),\mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})]$$

Theorem 7: Let $\mu^{\tilde{\mathcal{F}}}\in GFS(\mathcal{R})$be a generalized semistrongly convex fuzzy set. If there exists $\mu\in I^{1)}_{(0}$ for every $(\lambda\neq\gamma)$ $\lambda,\gamma\in{}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$ implies that

$$\mu^{\tilde{\mathcal{F}}}_{(\mu\lambda+(1-\mu)\gamma)}>[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma),\mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})],$$

then$\mu^{\tilde{\mathcal{F}}}$ is a generalized strongly convex fuzzy set on $\mathcal{R}$.

Proof: The proof is divided into four cases,

Case 1.If $\mu(\tilde{\mathcal{F}},\lambda)\neq\mu(\tilde{\mathcal{F}},\gamma)$, $\mu(\lambda,\tilde{\mathcal{F}})\neq\mu(\gamma,\tilde{\mathcal{F}})$ for every $(\lambda\neq\gamma)$ $\lambda,\gamma\in{}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$,then the proof is quickly follows from the hypothesis.

Case 2.If $\mu(\tilde{\mathcal{F}},\lambda)=\mu(\tilde{\mathcal{F}},\gamma)$, $\mu(\lambda,\tilde{\mathcal{F}})=\mu(\gamma,\tilde{\mathcal{F}})$ for every $(\lambda\neq\gamma)$ $\lambda,\gamma\in{}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$. From the hypothesis, we have

$$\mu(\tilde{\mathcal{F}},\mu\lambda+(1-\mu)\gamma)>\mu(\tilde{\mathcal{F}},\lambda)$$

and

$$\mu(\mu\lambda+(1-\mu)\gamma,\tilde{\mathcal{F}})>\mu(\gamma,\tilde{\mathcal{F}})$$

Assume that $\mu\lambda+(1-\mu)\gamma=\ell$ and ifϑ is any number in $I^{1)}_{(0}$, then we have three subcases

Subcase 1. If $\vartheta<\mu$, then for some $\sigma\in I^{1)}_{(0}$ we have

$$\mu(\tilde{\mathcal{F}},\vartheta\lambda+(1-\vartheta)\gamma)=\mu(\tilde{\mathcal{F}},\sigma\lambda+(1-\sigma)\ell)>\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\ell)$$
$$=\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\mu\lambda+(1-\mu)\gamma)>\mu(\tilde{\mathcal{F}},\lambda)=\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma)$$

and

$$\mu(\vartheta\lambda+(1-\vartheta)\gamma,\tilde{\mathcal{F}})=\mu(\sigma\lambda+(1-\sigma)\ell,\tilde{\mathcal{F}})>\mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\ell,\tilde{\mathcal{F}})$$
$$=\mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\mu\lambda+(1-\mu)\gamma,\tilde{\mathcal{F}})>\mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})$$

Subcase 2. If $\vartheta>\mu$, then for some $\delta\in I^{1)}_{(0}$ we have

$$\mu(\tilde{\mathcal{F}},\vartheta\lambda+(1-\vartheta)\gamma)=\mu(\tilde{\mathcal{F}},\sigma\ell+(1-\sigma)\gamma)>\mu(\tilde{\mathcal{F}},\ell)\wedge\mu(\tilde{\mathcal{F}},\gamma)$$
$$=\mu(\tilde{\mathcal{F}},\mu\lambda+(1-\mu)\gamma)\wedge\mu(\tilde{\mathcal{F}},\gamma)>\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma)$$

and

$$\mu(\vartheta\lambda+(1-\vartheta)\gamma,\tilde{\mathcal{F}})=\mu(\sigma\ell+(1-\sigma)\gamma,\tilde{\mathcal{F}})>\mu(\ell,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})$$
$$=\mu(\mu\lambda+(1-\mu)\gamma,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})>\mu(\gamma,\tilde{\mathcal{F}})=\mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})$$

Subcase 3. If $\vartheta=\mu$, then the proof is trivial.

Case 3.If $\mu(\tilde{\mathcal{F}},\lambda) \neq \mu(\tilde{\mathcal{F}},\gamma)$, $\mu(\lambda,\tilde{\mathcal{F}}) = \mu(\gamma,\tilde{\mathcal{F}})$ for every $(\lambda \neq \gamma)$ $\lambda,\gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$. From the hypothesis of the theorem, lower fuzzy set about $\mu^{\tilde{\mathcal{F}}}$ is a strongly convex fuzzy set and from the argument of case 2, upper fuzzy set about $\mu^{\tilde{\mathcal{F}}}$ is a strongly convex fuzzy set.

Case 4.If $\mu(\tilde{\mathcal{F}},\lambda) = \mu(\tilde{\mathcal{F}},\gamma)$, $\mu(\lambda,\tilde{\mathcal{F}}) \neq \mu(\gamma,\tilde{\mathcal{F}})$ for every $(\lambda \neq \gamma)$ $\lambda,\gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$. From the results of case 2, lower fuzzy set about $\mu^{\tilde{\mathcal{F}}}$ is a strongly convex fuzzy set and from the hypothesis, upper fuzzy set about $\mu^{\tilde{\mathcal{F}}}$ is a strongly convex fuzzy set.

Theorem 8: Let $\mu^{\tilde{\mathcal{F}}} \in GFS(\mathcal{R})$ be a generalized semistrongly convex fuzzy set. If ${}^{+}\mu^{\tilde{\mathcal{F}}}_{[\beta_1,\beta_2]}$ is closed for every $(\beta_1,\beta_2) \in I^{1]}_{[0} \times I^{1]}_{[0}$, then $\mu^{\tilde{\mathcal{F}}}$ is a generalized convex fuzzy set on $\mathcal{R}$.

Proof: For every $\lambda,\gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$ we have four cases:

Case 1.If $\mu(\tilde{\mathcal{F}},\lambda) \neq \mu(\tilde{\mathcal{F}},\gamma)$, $\mu(\lambda,\tilde{\mathcal{F}}) \neq \mu(\gamma,\tilde{\mathcal{F}})$, then the proof is trivial.

Case 2. If $\mu(\tilde{\mathcal{F}},\lambda) = \mu(\tilde{\mathcal{F}},\gamma)$and $\mu(\lambda,\tilde{\mathcal{F}}) = \mu(\gamma,\tilde{\mathcal{F}})$. Assume that, there exists $\delta \in I^{1]}_{[0}$ such that for $\delta\lambda + (1-\delta)\gamma = \Delta$

$$\begin{aligned}\mu^{\tilde{\mathcal{F}}}_{(\Delta)} &< \left[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma), \mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})\right] \\ &= \left[\mu(\tilde{\mathcal{F}},\lambda), \mu(\lambda,\tilde{\mathcal{F}})\right] \\ &= \left[\mu(\tilde{\mathcal{F}},\gamma), \mu(\gamma,\tilde{\mathcal{F}})\right]\end{aligned}$$

This implies that, for any $\sigma \in I^{1]}_{[0}$

$$\begin{aligned}\mu^{\tilde{\mathcal{F}}}_{(\sigma\lambda+(1-\sigma)\Delta)} &> \left[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\Delta), \mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\Delta,\tilde{\mathcal{F}})\right] \\ &= \left[\mu(\tilde{\mathcal{F}},\Delta), \mu(\Delta,\tilde{\mathcal{F}})\right]\end{aligned}$$

Since $\mu^{\tilde{\mathcal{F}}}$ is a generalized closed fuzzy set, there exists$\nu \in I^{1]}_{[0}$ such that for $\nu\lambda + (1-\nu)\Delta = \omega$,

$$\left[\mu(\tilde{\mathcal{F}},\Delta), \mu(\Delta,\tilde{\mathcal{F}})\right] < \mu^{\tilde{\mathcal{F}}}_{(\omega)} < \left[\mu(\tilde{\mathcal{F}},\gamma), \mu(\gamma,\tilde{\mathcal{F}})\right]$$

From this inequality and the semistrong convexity of $\mu^{\tilde{\mathcal{F}}}$. Let $\Delta = \eta\omega + (1-\eta)\gamma$ for some $\eta \in I^{1]}_{[0}$, we get

$$\mu^{\tilde{\mathcal{F}}}_{(\Delta)} > \left[\mu(\tilde{\mathcal{F}},\omega)\wedge\mu(\tilde{\mathcal{F}},\gamma), \mu(\omega,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})\right] > \mu^{\tilde{\mathcal{F}}}_{(\omega)}$$

Case 3.If $\mu(\tilde{\mathcal{F}},\lambda) = \mu(\tilde{\mathcal{F}},\gamma)$, $\mu(\lambda,\tilde{\mathcal{F}}) \neq \mu(\gamma,\tilde{\mathcal{F}})$. From the argument of case two, lower fuzzy set about $\mu^{\tilde{\mathcal{F}}}$ is a convex fuzzy set and from the hypothesis of the theorem, upper fuzzy set about $\mu^{\tilde{\mathcal{F}}}$ is also a convex fuzzy set.

Case 4.If $\mu(\tilde{\mathcal{F}},\lambda) \neq \mu(\tilde{\mathcal{F}},\gamma)$, $\mu(\lambda,\tilde{\mathcal{F}}) = \mu(\gamma,\tilde{\mathcal{F}})$. From the hypothesis, lower fuzzy set about $\mu^{\tilde{\mathcal{F}}}$ is a convex fuzzy set and from the results of case two, upper fuzzy set about $\mu^{\tilde{\mathcal{F}}}$ is also a convex fuzzy set.

Theorem 9: Let $\mu^{\tilde{\mathcal{F}}} \in GFS(\mathcal{R})$ and ${}^{+}\mu^{\tilde{\mathcal{F}}}_{[\beta_1,\beta_2]}$ is closed for every $(\beta_1,\beta_2) \in I^{1]}_{[0} \times I^{1]}_{[0}$. If there exists $\mu \in I^{1]}_{[0}$ for every $\lambda,\gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$, $\mu(\tilde{\mathcal{F}},\lambda) \neq \mu(\tilde{\mathcal{F}},\gamma)$, $\mu(\lambda,\tilde{\mathcal{F}}) \neq \mu(\gamma,\tilde{\mathcal{F}})$ implies that

$$\mu^{\tilde{\mathcal{F}}}_{(\mu\lambda+(1-\mu)\gamma)} > \left[\mu(\tilde{\mathcal{F}},\lambda)\wedge\mu(\tilde{\mathcal{F}},\gamma), \mu(\lambda,\tilde{\mathcal{F}})\wedge\mu(\gamma,\tilde{\mathcal{F}})\right],$$

then$\mu^{\tilde{\mathcal{F}}}$ is a generalized convex fuzzy set on $\mathcal{R}$.

Proof: Assume that, there exist $\alpha,\beta \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$ and $\varepsilon \in I^{1]}_{[0}$ for which

$$\mu^{\tilde{\mathcal{F}}}_{(\varepsilon\alpha+(1-\varepsilon)\beta)} < \left[\mu(\tilde{\mathcal{F}},\alpha)\wedge\mu(\tilde{\mathcal{F}},\beta), \mu(\alpha,\tilde{\mathcal{F}})\wedge\mu(\beta,\tilde{\mathcal{F}})\right]$$

In case of $\mu(\tilde{\mathcal{F}},\alpha) \neq \mu(\tilde{\mathcal{F}},\beta)$, $\mu(\alpha,\tilde{\mathcal{F}}) \neq \mu(\beta,\tilde{\mathcal{F}})$ with the hypothesis we get a contradiction. So that, $\mu(\tilde{\mathcal{F}},\alpha) = \mu(\tilde{\mathcal{F}},\beta)$, $\mu(\alpha,\tilde{\mathcal{F}}) = \mu(\beta,\tilde{\mathcal{F}})$ implies that

$$\left[\mu(\tilde{\mathcal{F}},\varepsilon\alpha + (1-\varepsilon)\beta), \mu(\varepsilon\alpha + (1-\varepsilon)\beta,\tilde{\mathcal{F}})\right] < \mu^{\tilde{\mathcal{F}}}_{(\beta)}$$

Let $(\sigma,\tau) \in I^{1]}_{[0.5} \times I^{0.5]}_{[0}$ such that $\theta = \sigma\alpha + \tau\beta$ and $\varphi = \tau\alpha + \sigma\beta$ where $\sigma = (1+\mu)^{-1}$and $\tau = \mu(1+\mu)^{-1}$. Implies that $\mu\varphi + (1-\mu)\alpha = \theta$and $\mu\theta + (1-\mu)\beta = \varphi$. Then

$$[\mu(\tilde{\mathcal{F}}, \mu\varphi + (1-\mu)\alpha), \mu(\mu\varphi + (1-\mu)\alpha, \tilde{\mathcal{F}})] > \mu^{\tilde{\mathcal{F}}}_{(\varphi)} \wedge \mu^{\tilde{\mathcal{F}}}_{(\alpha)}$$
$$= \mu^{\tilde{\mathcal{F}}}_{(\mu\theta+(1-\mu)\beta)} \wedge \mu^{\tilde{\mathcal{F}}}_{(\alpha)}$$
$$> \mu^{\tilde{\mathcal{F}}}_{(\theta)} \wedge \mu^{\tilde{\mathcal{F}}}_{(\beta)} \wedge \mu^{\tilde{\mathcal{F}}}_{(\alpha)}$$
$$= \mu^{\tilde{\mathcal{F}}}_{(\mu\varphi+(1-\mu)\alpha)} \wedge \mu^{\tilde{\mathcal{F}}}_{(\beta)}$$
$$= \mu^{\tilde{\mathcal{F}}}_{(\mu\varphi+(1-\mu)\alpha)} \wedge \mu^{\tilde{\mathcal{F}}}_{(\alpha)}$$
$$> \mu^{\tilde{\mathcal{F}}}_{(\mu\varphi+(1-\mu)\alpha)} \wedge \mu^{\tilde{\mathcal{F}}}_{(\mu\varphi+(1-\mu)\alpha)}$$

This implies that $\mu^{\tilde{\mathcal{F}}}_{(\mu\varphi+(1-\mu)\alpha)} > \mu^{\tilde{\mathcal{F}}}_{(\mu\varphi+(1-\mu)\alpha)}$ a contradiction.

In case of $\mu(\tilde{\mathcal{F}}, \alpha) \neq \mu(\tilde{\mathcal{F}}, \beta)$ and $\mu(\alpha, \tilde{\mathcal{F}}) = \mu(\beta, \tilde{\mathcal{F}})$ or $\mu(\tilde{\mathcal{F}}, \alpha) = \mu(\tilde{\mathcal{F}}, \beta)$and $\mu(\alpha, \tilde{\mathcal{F}}) \neq \mu(\beta, \tilde{\mathcal{F}})$. The above argument is repeated so it is omitted.

Theorem 10: If$\mu^{\tilde{\mathcal{F}}} \in GFS(\mathcal{R})$, then $\mu^{\tilde{\mathcal{F}}}$ is generalized strongly convex fuzzy set (resp. generalized semistrongly convex fuzzy set) if and only if lower fuzzy set and upper fuzzy set about $\mu^{\tilde{\mathcal{F}}}$ are strongly convex fuzzy set (resp. semistrongly convex fuzzy set).

Proof: The proof follows from the fact that

$$\mu^{\tilde{\mathcal{F}}}_{(\mu\lambda+(1-\mu)\gamma)} = [\mu(\tilde{\mathcal{F}}, \mu\lambda + (1-\mu)\gamma), \mu(\mu\lambda + (1-\mu)\gamma, \tilde{\mathcal{F}})]$$

Theorem 11: Let $\mu^{\tilde{\mathcal{F}}} \in GFS(\mathcal{R})$ is generalized strongly convex fuzzy set. If there exists $\mu \in I^{1)}_{(0}$ for every $\lambda, \gamma \in {}^{+}\mu^{\tilde{\mathcal{F}}}_{[0]}$, $\mu(\tilde{\mathcal{F}}, \lambda) \neq \mu(\tilde{\mathcal{F}}, \gamma)$, $\mu(\lambda, \tilde{\mathcal{F}}) \neq \mu(\gamma, \tilde{\mathcal{F}})$ implies that

$$\mu^{\tilde{\mathcal{F}}}_{(\mu\lambda+(1-\mu)\gamma)} > [\mu(\tilde{\mathcal{F}}, \lambda) \wedge \mu(\tilde{\mathcal{F}}, \gamma), \mu(\lambda, \tilde{\mathcal{F}}) \wedge \mu(\gamma, \tilde{\mathcal{F}})],$$

then$\mu^{\tilde{\mathcal{F}}}$ is a generalized semistrongly convex fuzzy set on $\mathcal{R}$.

Proof: It is easy to see that generalized strongly convex fuzzy set is generalized convex fuzzy set. So the result follows immediately from the argument of Theorem 10.

Remark 12: From the relation of α-cut and definition of fuzzy convex set. One can study strong relations among generalized strongly convex fuzzy sets, generalized semistrongly convex fuzzy sets, ${}^{+}\mu^{\tilde{A}}_{[\beta_1,\beta_2]}$, and ${}^{-}\mu^{\tilde{A}}_{(\beta_1,\beta_2)}$.

Conclusion

In this paper, we defineda new version of fuzzy convexity such as generalized convex fuzzy sets, generalized strongly convex fuzzy sets, and generalized semistrongly convex fuzzy sets and found some important connections among them.Also, the intersection and union operations on them are illustrated and examples are given for the validity of the decision. We have demonstrated some conditions for which generalized convex fuzzy sets become generalized semistrongly convex fuzzy sets and found a simpler condition for vice versa. Furthermore, we declared relations between generalized semistrongly convex fuzzy sets and generalized strongly convex fuzzy sets, and some conditions for which generalized fuzzy sets become generalized convex fuzzy setsare given. Finally, we presented relationship between these versions of fuzzy convexity withthe lower and upper fuzzy sets.

Acknowledgments

The authoris grateful to the anonymous referees for their valuable comments and suggestions.The author also wishes to thank theScientific Committee of Mathematics Department and Scientific Committee of Faculty of Sciences and Education Sciences for facilitating the project.

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